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Reflection of plane waves at the free surface of a fibre-reinforced elastic half-space

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Abstract. The propagation of plane waves in fibre-reinforced, anisotropic, elastic media is discussed. The expressions for the phase velocity of quasi-P (qP) and quasi-SV (qSV) waves propagating in a plane containing the reinforcement direction are obtained as functions of the angle between the propagation and reinforcement directions. Closed form expressions for the amplitude ratios for qP and qSV waves reflected at the free surface of a fibre-reinforced, anisotropic, homogeneous, elastic half-space are obtained. These expressions are used to study the variation of the amplitude ratios with angle of incidence. It is found that the reinforcement has a significant effect on the amplitude ratios and the critical angle.

Keywords. Anisotropic medium; fibre-reinforced medium; half-space; phase velocity; reflection.

1. Introduction

Fibre-reinforced composites are used in a variety of structures due to their low weight and high strength. In most previous investigations on the reflection of waves at the free surface of an elastic half-space, the effect of reinforcement has been neglected. The characteristic property of a reinforced composite is that its components act together as a single anisotropic unit as long as they remain in the elastic condition. Sengupta & Nath (2001) discussed the problem of surface waves in fibre-reinforced anisotropic elastic media. They expressed the plane strain displacement components in terms of two scalar potentials to decouple the plane motion into qP and qSV waves. Singh (2002) showed that, for wave propagation in fibre-reinforced anisotropic media, this decoupling cannot be achieved by the introduction of the displacement potentials. In the present paper, the problem of the reflection of qP and qSV waves at the free surface of a fibre-reinforced anisotropic elastic half-space is studied by a direct method without the introduction of potentials. Reflection of plane wave at the free surface of an anisotropic half-space has been studied, among others, by Ditri & Rose (1992), Zilmer *et al* (1997) and Singh & Khurana (2002). Hashin & Rosen (1964) gave the elastic moduli for fibre-reinforced materials.

2. Basic equations

The constitutive equations for a fibre-reinforced linearly elastic anisotropic medium with respect to the reinforcement direction **a** are (Belfield *et al* 1983)

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha(a_k a_m e_{km}\delta_{ij} + e_{kk}a_i a_j) \\ + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + \beta a_k a_m e_{km} a_i a_j, \quad (1)$$

where σ_{ij} are the components of stress; e_{ij} are the components of strain; λ, μ_T are elastic constants; α, β, $(\mu_L - \mu_T)$ are reinforcement parameters and $\mathbf{a} = (a_1, a_2, a_3)$; $a_1^2 + a_2^2 + a_3^2 = 1$. We choose the fibre-direction as $\mathbf{a} = (1, 0, 0)$. The strains can be expressed in terms of the displacements u_i as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For plane strain deformation in the x_1x_2-plane, $\partial/\partial x_3 \equiv 0$, $u_3 = 0$. Equation (1) then yields

$$\sigma_{11} = A_{11}\frac{\partial u_1}{\partial x_1} + A_{12}\frac{\partial u_2}{\partial x_2},$$

$$\sigma_{22} = A_{12}\frac{\partial u_1}{\partial x_1} + A_{22}\frac{\partial u_2}{\partial x_2},$$

$$\sigma_{33} = A_{12}\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2},$$

$$\sigma_{12} = \mu_L\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right),$$

$$\sigma_{31} = \sigma_{32} = 0, \quad (2)$$

where

$$A_{11} = \lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta,$$

$$A_{12} = \lambda + \alpha, \quad A_{22} = \lambda + 2\mu_T. \quad (3)$$

The equations of motion without body forces are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2}(i = 1, 2, 3), \quad (4)$$

using the summation convention. From (2) we note that the third equation of motion in (4) is identically satisfied and the first two equations become

$$A_{11}\frac{\partial^2 u}{\partial x^2} + B_2\frac{\partial^2 v}{\partial x \partial y} + B_1\frac{\partial^2 u}{\partial y^2} = \rho\frac{\partial^2 u}{\partial t^2}, \quad (5)$$

$$A_{22}\frac{\partial^2 v}{\partial y^2} + B_2\frac{\partial^2 u}{\partial x \partial y} + B_1\frac{\partial^2 v}{\partial x^2} = \rho\frac{\partial^2 v}{\partial t^2}, \quad (6)$$

where we have used the notation $x_1 = x$, $x_2 = y$, $u_1 = u$, $u_2 = v$, $B_1 = \mu_L$, $B_2 = \alpha + \lambda + \mu_L$.

3. Propagation of plane waves

Consider a fibre-reinforced, homogeneous, elastic medium occupying the half-space $y \geq 0$. For plane waves of circular frequency ω, wave number k and phase velocity c, propagating in the xy-plane and incident at the free boundary $y = 0$ at an angle θ with the y-axis (figure 1), we may assume

$$u = U \exp(iP_1), \quad v = V \exp(iP_1), \tag{7}$$

where U, V are the amplitude factors and,

$$P_1 = \omega t - k(x \sin\theta - y \cos\theta),$$

is the phase factor. For waves reflected at $y = 0$, we assume

$$u = U \exp(iP_2), \quad v = V \exp(iP_2), \tag{8}$$

where

$$P_2 = \omega t - k(x \sin\theta + y \cos\theta)$$

is the phase factor associated with reflected waves. Making use of (7) or (8) in (5) and (6), we obtain

$$-(D_1 - \rho c^2)U \pm B_2 \sin\theta \cos\theta V = 0, \tag{9}$$

$$\pm B_2 \sin\theta \cos\theta U - (D_2 - \rho c^2)V = 0, \tag{10}$$

where the upper sign corresponds to the incident waves and the lower sign corresponds to the reflected waves. D_1, D_2 are given by

$$D_1(\theta) = A_{11} \sin^2\theta + B_1 \cos^2\theta,$$

$$D_2(\theta) = A_{22} \cos^2\theta + B_1 \sin^2\theta. \tag{11}$$

Equations (9) and (10) can have a nontrivial solution only if

$$\begin{vmatrix} -(D_1 - \rho c^2) & \pm B_2 \sin\theta cos\theta \\ \pm B_2 \sin\theta \cos\theta & -(D_2 - \rho c^2) \end{vmatrix} = 0. \tag{12}$$

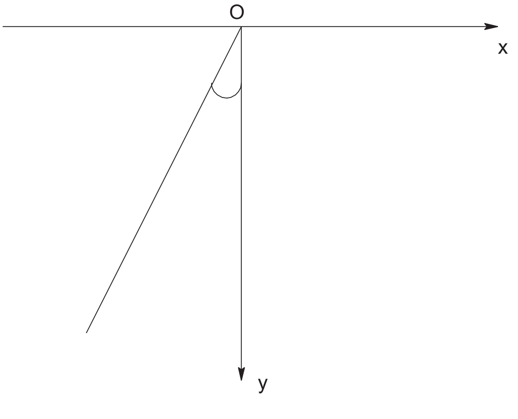

Figure 1. Geometry of the problem.

The solutions of the quadratic equation (12) in ρc^2 are

$$2\rho c^2(\theta) = (D_1 + D_2) \pm [(D_1 - D_2)^2 + 4B_2^2 \sin^2\theta \cos^2\theta]^{1/2}. \tag{13}$$

Thus, in this two-dimensional model of the fibre-reinforced anisotropic medium, there are two types of plane waves whose phase velocities depend on the angle of incidence θ. Let $c_1(\theta)$ and $c_2(\theta)$ be the values of c associated with the upper and the lower signs respectively, in (13). On neglecting reinforcement parameters, c_1 and c_2 reduce to the usual velocities of P and S waves in an isotropic material with Lame constants λ, μ_T and density ρ. We call the waves with velocities c_1 and c_2 as quasi-P and quasi-S waves respectively. We next discuss the reflection of these waves at the free boundary $y = 0$.

4. Quasi-P waves incident at the free boundary

If quasi-P waves are incident at the boundary $y = 0$ of the fibre-reinforced anisotropic semi-infinite medium, we get quasi-P and quasi-SV waves as reflected waves. We may, therefore, assume the total displacement to be of the form

$$u = U_0 \exp(iR_1) + U_1 \exp(iS_1) + U_2 \exp(iS_2), \tag{14}$$

$$v = V_0 \exp(iR_1) + V_1 \exp(iS_1) + V_2 \exp(iS_2), \tag{15}$$

where

$$\begin{aligned} R_1 &= (\omega/c_1)[c_1 t - (x \sin e - y \cos e)],\\ S_1 &= (\omega/c_1)[c_1 t - (x \sin e + y \cos e)],\\ S_2 &= (\omega/c_2)[c_2 t - (x \sin f + y \cos f)], \end{aligned} \tag{16}$$

are the phase factors associated with the incident quasi-P, reflected quasi-P and reflected quasi-SV waves, e being the angle which incident and reflected quasi-P waves make with y-axis and f is the angle which the reflected quasi-SV waves make with the y-axis. U_0, V_0; U_1, V_1; and U_2, V_2 are the amplitude factors associated with the incident quasi-P, reflected quasi-P and reflected quasi-SV waves respectively. Since the incident and reflected waves in (14) and (15) must satisfy the equations of motion (5) and (6), we have, as in (9),

$$\begin{aligned} -[D_1(e) - \rho c_1^2(e)]U_0 + B_2 \sin e \cos e V_0 &= 0,\\ -[D_1(e) - \rho c_1^2(e)]U_1 - B_2 \sin e \cos e V_1 &= 0,\\ -[D_1(f) - \rho c_2^2(f)]U_2 - B_2 \sin f \cos f V_2 &= 0. \end{aligned} \tag{17}$$

It may be noted that we can obtain another set of similar equations corresponding to (10). But this set will give the same results as the set in (17) due to consistency condition (12). Equations (17) may be written as

$$U_0 = \eta_1 V_0, \qquad U_1 = -\eta_1 V_1, \qquad U_2 = -\eta_2 V_2, \tag{18}$$

where

$$\begin{aligned} \eta_1 &= B_2 \sin e \cos e/[D_1(e) - \rho c_1^2(e)],\\ \eta_2 &= B_2 \sin f \cos f/[D_1(f) - \rho c_2^2(f)]. \end{aligned} \tag{19}$$

The total displacement field in (14) and (15) must satisfy the boundary conditions,

$$\sigma_{21} = \sigma_{22} = 0, \quad \text{at} \quad y = 0. \tag{20}$$

Making use of (2), (14) and (15) in the above boundary conditions, we obtain

$$\begin{aligned}
&[(\cos e/c_1)U_0 + (-\sin e/c_1)V_0]\exp[iS_1(x,0)] \\
&\quad + [(-\cos e/c_1)U_1 + (-\sin e/c_1)V_1]\exp[iS_1(x,0)] \\
&\quad + [(-\cos f/c_2)U_2 + +(-\sin f/c_2)V_2]\exp[iS_2(x,0)], \qquad (21)\\
&[(\lambda+\alpha)(-\sin e/c_1)U_0 + (\lambda+2\mu_T)(\cos e/c_1)V_0]\exp[iS_1(x,0)] \\
&\quad + [(\lambda+\alpha)(-\sin e/c_1)U_1 + (\lambda+2\mu_T)(-\cos e/c_1)V_1]\exp[iS_1(x,0)] \\
&\quad + [(\lambda+\alpha)(-\sin f/c_2)U_2 + (\lambda+2\mu_T)(-\cos f/c_2)V_2]\exp[iS_2(x,0)], \qquad (22)
\end{aligned}$$

where

$$c_1 = c_1(e), \quad c_2 = c_2(f), \tag{23}$$

and where we have used the result

$$R_1(x,0) = S_1(x,0). \tag{24}$$

Since (21) and (22) must be satisfied for all values of x, we have

$$S_1(x,0) = S_2(x,0), \tag{25}$$

which, on using (16), gives

$$\sin e/c_1(e) = \sin f/c_2(f). \tag{26}$$

This is the form of Snell's law for the fibre-reinforced anisotropic medium. Equations (21) and (22), with the help of (18) and (25), may be written as

$$m_1 V_0 + m_1 V_1 + m_2 V_2 = 0, \tag{27}$$

$$n_1 V_0 - n_1 V_1 + n_2 V_2 = 0, \tag{28}$$

where

$$\begin{aligned}
m_1 &= (\eta_1 \cos e - \sin e)/c_1, \\
m_2 &= (\eta_2 \cos f - \sin f)/c_2, \\
n_1 &= [(\lambda+\alpha)\eta_1 \sin e - (\lambda+2\mu_T)\cos e]/c_1, \\
n_2 &= -[(\lambda+\alpha)\eta_2 \sin f - (\lambda+2\mu_T)\cos f]/c_2.
\end{aligned} \tag{29}$$

The amplitude ratios for the reflected waves are obtained from (18), (27) and (28) as

$$U_1/U_0 = (m_1 n_2 - m_2 n_1)/\Delta, \quad U_2/U_0 = 2\eta_2 m_1 n_1/\eta_1 \Delta, \tag{30}$$

$$V_1/V_0 = (m_2 n_1 - m_1 n_2)/\Delta, \quad V_2/V_0 = -2 m_1 n_1/\Delta, \tag{31}$$

where

$$D = m_1 n_2 + m_2 n_1. \tag{32}$$

Equation (30) gives the amplitude ratios for the horizontal component of the displacement. Similarly, (31) gives the amplitude ratios for the vertical component. From (30) and (31), we obtain the following expressions for the amplitude ratios for the total displacement for reflected qP and qSV waves when qP waves are incident at the free boundary

$$Z_1^{PP} = |(m_2 n_1 - m_1 n_2)/\Delta|, \tag{33}$$

$$Z_2^{PS} = \|2m_1 n_1/\Delta\|[(\eta_2^2 + 1)/(\eta_1^2 + 1)]^{1/2}. \tag{34}$$

5. Quasi-SV waves incident at the free boundary

We assume that the incident and reflected quasi-SV waves make an angle f with the y-axis and the reflected quasi-P waves make an angle e with this axis. Amplitude ratios in this case may be obtained as in the previous section. We find

$$U_1/U_0 = 2\eta_1 m_2 n_2/\eta_2 \Delta, \quad U_2/U0 = (m_2 n_1 - m_1 n_2)/\Delta, \tag{35}$$

$$V_1/V_0 = -2m_2 n_2/\Delta, \quad V_2/V_0 = (m_1 n_2 - m_2 n_1)/\Delta, \tag{36}$$

where U_0, V_0 are the amplitude factors of the incident quasi-SV waves.

From (35) and (36), we obtain the amplitude ratios for the total displacement for reflected qP and qSV waves as

$$Z_1^{SP} = \|2m_2 n_2/\Delta\|[(\eta_1^2 + 1)/(\eta_2^2 + 1)]^{1/2}, \tag{37}$$

$$Z_2^{SS} = |(m_1 n_2 - m_2 n_1)/\Delta|. \tag{38}$$

6. Numerical results and discussion

To study the effect of reinforcement on wave propagation, we use the following numerical values for the physical constants

$$\lambda = 7{\cdot}59 \times 10^{11}\ \text{dyne/cm}^2, \qquad \mu_T = 1{\cdot}89 \times 10^{11}\ \text{dyne/cm}^2,$$

$$\mu_L = 2{\cdot}45 \times 10^{11}\ \text{dyne/cm}^2, \qquad \alpha = -1{\cdot}28 \times 10^{11}\ \text{dyne/cm}^2,$$

$$\beta = 0{\cdot}32 \times 10^{11}\ \text{dyne/cm}^2, \qquad \rho = 7{\cdot}8\ \text{gm/cm}^3.$$

Making use of Snell's law given by (26), the angles of reflection for qP and qSV waves are computed for various values of the angle of incidence of qSV and qP waves respectively. Figure 2 gives the angle of reflection of qSV waves for various values of the angle of incidence of qP waves with and without reinforcement. For the present choice of the physical constants, the angle of reflection for qSV waves in the presence of reinforcement increases more rapidly than in the absence of reinforcement. Figure 3 gives the angle of reflection of qP waves for various values of the angle of incidence of qSV waves with and without reinforcement.

The amplitude ratios for reflected qP and qSV waves are computed for incident qP and qSV waves. The variations of these amplitude ratios with the angle of incidence are shown graphically in figures 4 and 5 for incident qP and qSV waves, respectively. The dotted lines with and without crosses represent the variations of the amplitude ratios for reflected waves in the absence of reinforcement. The solid lines with and without crosses represent the amplitude ratios with reinforcement.

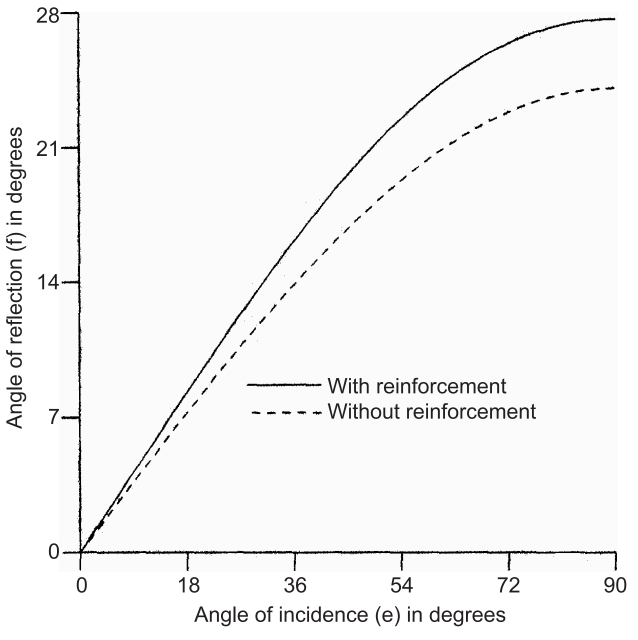

Figure 2. Variation of the angle of reflection of qSV waves with the angle of incidence of qP waves.

6.1 *Incident quasi-P wave*

The variations of the amplitude ratios for reflected qP and qSV waves with the angle of incidence of qP waves are shown in figure 4 by solid lines with crosses and solid lines, respectively. The amplitude ratio for reflected qP is one at $e = 0^\circ$ and $e = 90^\circ$ and attains

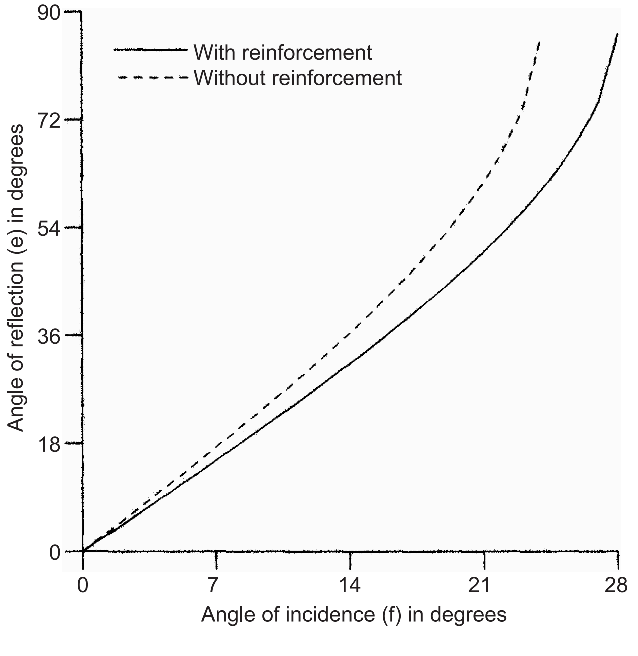

Figure 3. Variation of the angle of reflection of qP waves with the angle of incidence of qSV waves.

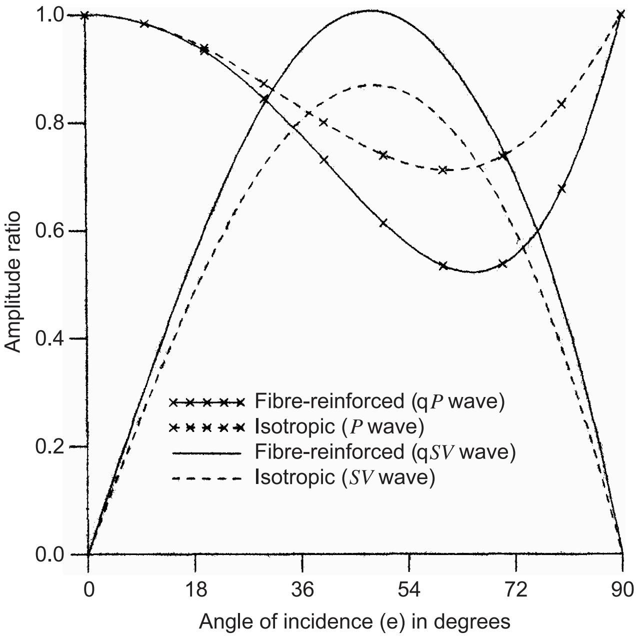

Figure 4. Variation of the amplitude ratios for reflected waves with the angle of incidence of qP waves.

its minimum at $e = 65^\circ$. Also, the amplitude ratio for reflected qSV has its value zero at $e = 0^\circ$ and $e = 90^\circ$ and attains its maximum at $e = 48^\circ$. A comparison between solid and dotted lines reveals that the effect of reinforcement on amplitude ratios of reflected qP and qSV waves is significant.

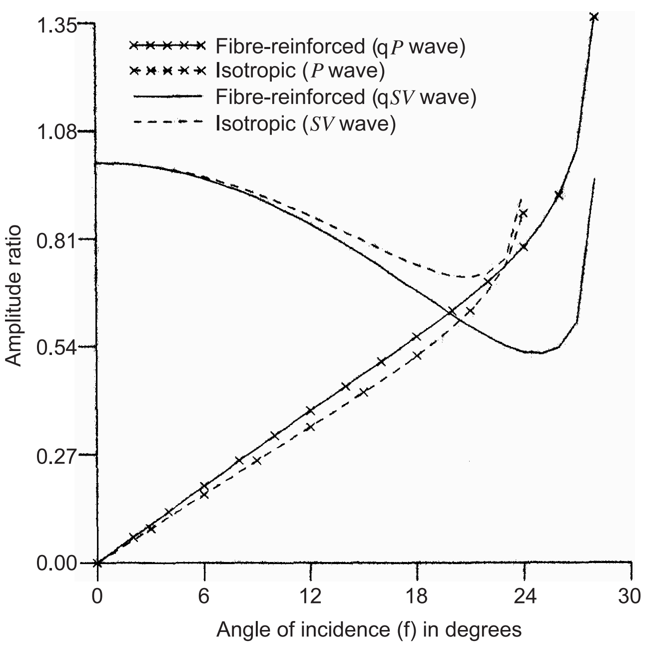

Figure 5. Variation of the amplitude ratios for reflected waves with the angle of incidence of qSV waves.

6.2 *Incident quasi-SV wave*

The variations of the amplitude ratios for reflected qP and qSV waves with the angle of incidence of qSV wave are shown in figure 5 by solid lines with crosses and solid lines respectively. The amplitude ratio for reflected qP is zero at $f = 0^\circ$ and it attains its maximum near $f = 28^\circ$. Beyond $f = 28^\circ$, it is zero for all angles of incidence. Thus the critical angle for reflected qP waves is 28°. Also, the amplitude ratio for reflected qSV wave is one at $f = 0^\circ$ and decreases to its minimum at $f = 25^\circ$. For the range $25^\circ \leq f \leq 28^\circ$, it increases. Beyond $f = 28^\circ$, it is one for all angles of incidence. The effect of reinforcement as defined by the physical constants chosen is to increase the critical angle from about 24° to about 28°.

7. Conclusions

Equations (33), (34), (37) and (38) give the amplitude ratios of the reflected waves when qP and qSV waves are incident at the free surface of a fibre-reinforced, homogeneous, anisotropic, elastic half-space. It is assumed that the plane of incidence contains the reinforcement direction. We have verified that, on putting the reinforcement parameters α, β and $(\mu_L - \mu_T)$ zero each, the amplitude ratios obtained in the present study coincide with the amplitude ratios for an isotropic half-space as given by Ben-Menahem & Singh (1981, pp 93–95). We have found that the reinforcement has a significant effect on the amplitude ratios. Further, the reinforcement alters the critical angle considerably.

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